Symmetry-Induced Tunnelling in One-Dimensional Disordered Potentials

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A new mechanism of tunnelling at macroscopic distances is proposed for a wave packet localized in one-dimensional disordered potential with mirror symmetry, $V(-x) = V(x)$. Unlike quantum tunnelling through a regular potential barrier, which occurs only at the energies lower then the barrier height, the proposed mechanism of tunnelling exists even for weak white-noise-like scattering potentials. It also exists in classical circuits of resonant contours with random resonant frequencies. The latter property may be used as a new method of secure communication, which does not require coding and decoding of the transmitting signal.

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It is well-known that all quantum states in one-dimensional white-noise potential are strongly localized and quantum transport is limited by the distances not exceeding the localization length $l(E)$. At longer distances the destructive interference between direct and backscattered waves suppresses exponentially the amplitude of a wave packet. Statistical correlations in the disordered potential may change the interference pattern and may give rise to a discrete set [1] or to a continuum of delocalized states [2, 3] for short- or long-range correlation respectively. Correlations is a manifestation of the local properties of a random potential. The symmetry is a global property, therefore its effect on the transport may be even stronger.

In this Letter we propose a symmetry-driven mechanism of tunnelling, which is specific

for the random potentials only. Usually, the symmetry is considered to be an irrelevant property in disordered systems since the wave functions are localized at the (local) scales, which are much smaller than the (global) scales, where the symmetry is manifested. However, the symmetry of the potential, $V(-x) = V(x)$, leads to definite parity of the wave functions. Either parity (even or odd) of an eigenfunction means that there are *two* equal peaks with half-width $\sim l(E)$ centered at the symmetric points. A symmetry-induced correlation between these peaks gives rise to the mechanism of tunnelling of a wave packet (or excitation), independently how far apart the peaks are. Due to this mechanism a wave packet tunnels at macroscopic distances – a process which does not exist for the random potential without the symmetry. Natural disorder usually does not exhibit the mirror symmetry. Nevertheless, the proposed mechanism of tunnelling is not of pure academic interest, since it may be observed also in a classical system – a random electrical circuit, where the symmetry can be easily introduced. In what follows we propose a new method of secure communications based on the symmetry-induced mechanism of tunnelling. The merit of this method is that it does not require a coding-decoding procedure.

To demonstrate the main idea of the symmetry-induced tunnelling we consider the tight-binding Anderson model [4]. For one-dimensional lattice a stationary solution for the eigenstate with energy E is obtained from the equation

$$\psi_{n+1} + \psi_{n-1} = (E + \epsilon_n)\,\psi_n\,, \tag{1}$$

where ϵ_n is on-site energy. The energies E and ϵ_n are measured in units of the hopping amplitude t, which in the case of diagonal disorder is independent on the site index n.

Discrete Schrodinger equation (1) gives *exact* description of the electrical circuit of classical impedances Z_n and z_n shown in Fig. 1. Application of Kirchhoff's Loop Rule to three successive unit cells of the circuit leads to the following linear relation between the currents circulating in the $(n-1)$-th, n-th and $(n+1)$-th cells

$$z_n I_{n+1} + z_{n-1} I_{n-1} = (Z_n + z_n + z_{n+1}) I_n\,. \tag{2}$$

If the vertical impedances are all the same, $z_n = z_0$, this equation is reduced to the tight-binding model with diagonal disorder, Eq. (1), with $\epsilon_n = \delta_n/z_0$ and $E = 2 + Z_0/z_0$. Here the random value of the impedance Z_n is split into its mean value $Z_0 = \langle Z_n\rangle$ and the fluctuating part $\delta_n = Z_n - Z_0$.

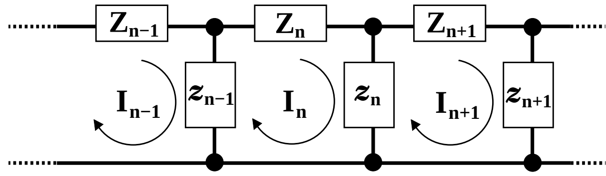

FIG. 1: Segment of infinite electric circuit of classical impedances.

This exact correspondence allows testing of quantum effects of Anderson localization using classical electrical circuits with random elements. In fact, during the last decade chaotic resonant cavities have been successfully used for testing the predictions of quantum chaos [5]. It is worth mentioning that electrical circuits of passive elements have been widely used for modelling different physical phenomena. The first application of the method of equivalent circuit probably goes back to Lord Kelvin who used a discrete RC chain to study a signal transmission through a transatlantic cable. Many bright examples of electrical circuits that model quantum mechanical behavior for simple but fundamental systems are given in the book by Pippard [6]. Recently it was proposed that electromagnetic waveguide can be used to model as exotic effect as Hawking black hole radiation [7]. Some effects of correlated disorder have been studied in the experiments with microwave propagation through disordered waveguides [8] and sub-terahertz response of superconducting multilayers [9]. Experimental realization of a system with desirable correlations and observation of the localized and extended states are much easier in electromagnetic devices [8, 9] than in heterostructures with intentionally introduced disorder [10].

If the potential in Eq. (1) is an even function, $\epsilon_n = \epsilon_{-n}$, the eigenfunctions Ψ_n^α are either even or odd functions of n. If an eigenfunction Ψ_n^α is localized near a site n_0, the amplitude of this state at the origin is exponentially suppressed, $\Psi_{n=0}^\alpha \propto \exp(- \mid n_0 \mid / l(E_\alpha))$, provided $\mid n_0 \mid \gg l(E_\alpha)$. However, due to definite parity of the wave function, another peak appears at the symmetric point $n = -n_0$. Strong localization of any excitation in random potential is a result of *destructive* interference between propagating and backscattered wave. The appearance of the symmetric peak can be explained as a result of *constructive* interference. It leads to exponential increase of the amplitude of the wave, i.e. to *antilocalization* [11].

In Fig. 2 we show two quasi-degenerate eigenstates calculated for the symmetric potential of 1000 sites (i.e. only 500 of these sites are random). The inverse localization length (the

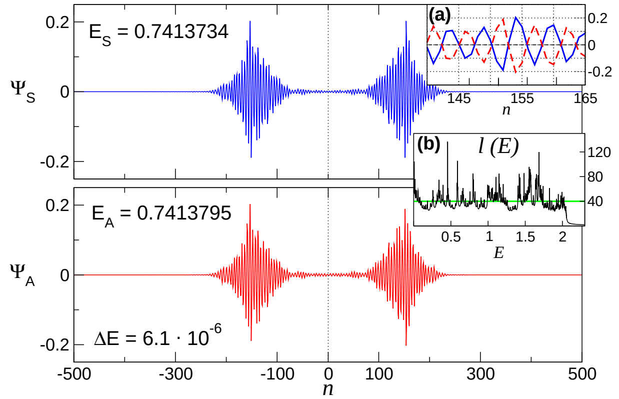

FIG. 2: (Color on line) Two eigenstates with different parity (Ψ_S is even and Ψ_A is odd) in random symmetric potential $\epsilon_{-n} = \epsilon_n$ with $\langle \epsilon_n \rangle = 0$ and $\langle \epsilon_n^2 \rangle = \epsilon_0^2 = 0.1$. These states belong to a doublet with energy splitting ΔE. Inserts: (a) Blow-up of the right peaks of the eigenfunctions showing that they possess different parity. (b) Numerical result for the localization length as compared to the energy-independent function $l(E) = 40$. The compensation of the energy dependence in $l_0(E)$ is not of principal importance and is done only to simplify the discussion of the numerical results.

Lyapunov exponent) can be estimated from the formula [3]

$$l^{-1}(E) = l_0^{-1}(E)\varphi(\mu), \;\; \varphi(\mu) = 1 + 2\sum_{k=1}^{\infty} \xi(k)\, \cos{(2\mu\, k)}. \tag{3}$$

Here $l_0^{-1}(E) = \epsilon_0^2/(8\sin^2\mu)$ is the Thouless [12] result for the white noise disorder, the function $\varphi(\mu)$ accounts for the contribution of correlations with correlation function $\langle \epsilon_n \epsilon_{n+k} \rangle = \epsilon_0^2 \xi(k)$, and the dispersion relation is $E = 2\cos\mu$. The results shown in Fig. 2 are obtained not for white-noise but for slightly correlated disorder with correlator $\xi(1) = -1/2$ and $\xi(k > 1) = 0$. These short-range correlations are introduced in order to compensate the smooth energy dependence of $l_0(E)$. It is easy to see that the contribution of the term with $k = 1$ in Eq. (3) provides a flat dependence $l^{-1}(E) = \epsilon_0^2/4 = const$. Insert (b) in Fig. 2 shows the numerical values of $l(E)$, which fluctuate around 40 – the value obtained from Eq. (3). In agrement with this estimate, the half-width of the peaks in Fig. 2 is approximately 40 sites.

The energy spectrum of Eq. (1) with symmetric random potential is similar to the spectrum of a double-well potential. It consists of discrete levels, most of them lying within the interval $-2 < E < 2$. The energy levels are arranged in doublets of states with different parity. The energy $\delta(E)$ between the centers of the doublets scales with the length of

the system N as $1/N$. The energy splitting ΔE in the doublet is exponentially small, $\Delta(E) \propto \exp[-4 \mid n_0 \mid /l(E)]$, i.e. the states are quasi-degenerate. Both, $\delta(E)$ and $\Delta(E)$ fluctuate with energy because of statistical fluctuations of the density of states, $n_0(E)$, and $l(E)$.

The symmetry-induced tunnelling can be observed in the dynamics of an excitation. Let a perturbation is applied at one of the sites of the symmetric random sequence. In the simplest case the perturbation is a δ-excitation at the site n_0, $\psi_n(t=0) = \delta_{nn_0}$. Since this excitation is not an eigenfunction of the system, its temporal evolution is represented as a superposition,

$$\psi_n(t) = \sum_\alpha C^\alpha_{n_0} \Psi^\alpha_n \exp(-iE_\alpha t). \tag{4}$$

The sum in Eq. (4) runs over the eigenstates, which are all localized. The eigenstates centered closer to the initial excitation contribute more because the coefficient $C^\alpha_{n_0} = \langle \Psi^\alpha_n | \psi_n(t=0) \rangle = \Psi^\alpha_{n_0}$ is the overlapping integral between the initial excitation and the eigenstate Ψ^α_n. Let the eigenstates with maximum overlapping be Ψ_A and Ψ_S. They form a doublet with the central energy $\bar{E} = (E_A + E_S)/2$ and splitting $\Delta E = E_A - E_S$. Taking into account only these two terms in Eq. (4), the following approximate result for the evolution of the initial excitation can be easily obtained

$$\psi_n(t) \approx \frac{e^{-i\bar{E}t}}{\sqrt{l(\bar{E})}} \left[\cos\left(\frac{\Delta E}{2} t\right) \Psi_+(n) + i \sin\left(\frac{\Delta E}{2} t\right) \Psi_-(n) \right]. \tag{5}$$

Here $\Psi_\pm(n) = (\Psi_S \pm \Psi_A)/\sqrt{2}$. Each of these linear combinations is a single-peak function. The peak of Ψ_+ is always close to the point of initial excitation. For the eigenfunctions shown in Fig. 2 the peak Ψ_+ is localized in the region of negative n.

At the early stage of evolution the initial δ-peak at n_0 quickly spreads over the region of width $2l(\bar{E})$. Further spreading is suppressed by Anderson localization. Eq. (5) is not valid at this transient stage. This equation describes steady and "slow" harmonic oscillations of the initial excitation between the two symmetrical points. The period of oscillations of the density $\mid \psi_n(t) \mid^2$ is $T = 2\pi/\Delta E$. If the distance $2 \mid n_0 \mid$ between the peaks exceeds the localization length, the amplitude of the wave function at the origin is exponentially small $\sim \exp(- \mid n_0 \mid /l(\bar{E}))$. It, however, grows exponentially towards the symmetrical point $-n_0$. This increase is a manifestation of the tunnelling induced by the symmetry. The dynamics of penetration of the initial excitation to the symmetrical point is very similar to the tunnelling

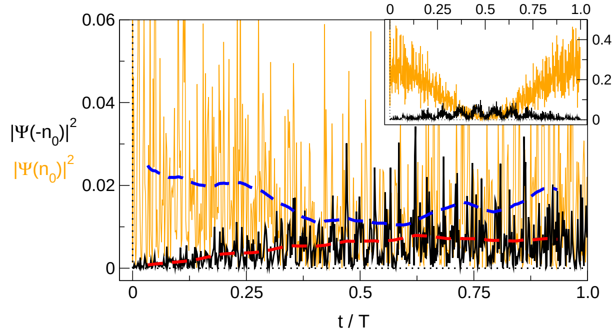

FIG. 3: (Color on line) Temporal evolution of the probability density $|\psi(t)|^2$ calculated at the point of excitation, $n_0 = -153$ (grey line) and at the symmetric point, $-n_0 = 153$ (black line). Dashed lines are the window-average values of the densities at the symmetric points and are drawn for guide-eye only. The insert shows similar temporal evolution but for much stronger disorder, $\epsilon_0^2 = 0.55$. In this case the localization length is shorter $l(E) = 8$ and much less doublets contribute to the evolution of the the initial excitation.

through a potential barrier, although there is no real barrier. Exponential decrease (increase) of the wave functions is due to multiple scattering events with predominant destructive (constructive) interference. One can speak about an effective double-well potential which produces the same discrete energy spectrum. Calculation of the parameters of this effective potential is a challenging inverse-scattering problem. Tunnelling processes without a real barrier are known in dynamical systems, where quantum transitions occur either between strongly localized states [13] or between classically separated regions in phase space [14]. It is worth mentioning that regular Bloch-like oscillations in a potential with correlated disorder may occur also due to the presence of two mobility edges in the energy spectrum [15].

Eq. (5) takes into account interaction of the initial excitation with the nearest doublet. If there are more eigenstates in Eq. (4), whose wave functions extend to the point n_0, they also contribute to the evolution of the initial excitation. In this case the oscillations between the peaks at n_0 and $-n_0$ are not harmonic any more but a superposition of harmonics with different periods. In the numerical study of evolution of the excitation we take into account its interaction with the eigenfunctions which have amplitude $> 10^{-3}$ at the site $n_0 = -153$. There are 540 such eigenfunctions out of total 1000. These eigenfunctions produce the oscillatory pattern in Fig. 3. Although the site $n_0 = -153$ is the position of the maximum for the eigenstates Ψ_A and Ψ_S in Fig. 2, other states give a noticeable contribution. Since

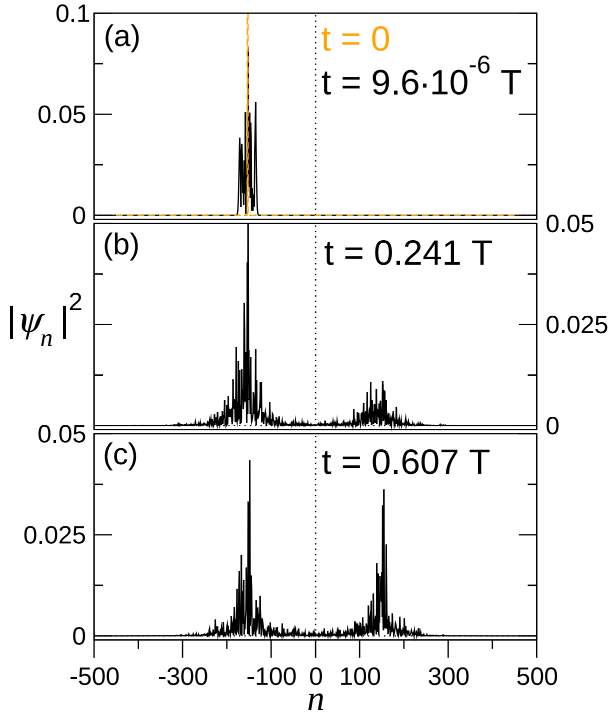

FIG. 4: (Color on line) Spatial distribution of the probability $\mid \psi_n(t) \mid^2$ at three instants: (a) Spreading of the initial peak at the transient stage, $t = 9.6 \cdot 10^{-6} T$. The initial peak of amplitude 1 at $n_0 = -153$ is shown in grey. The secondary peak is not visible at this stage. (b) The secondary peak at $-n_0 = 153$ is well developed at $t = 0.241T$; (c) The two peaks become almost equal at $t = 0.607T$.

the levels splittings in different doublets are random and incommensurate, the evolution of the wave packet is not periodic but it keeps the main features predicted by Eq. (5). In the case of stronger localization of the eigenstates the dependence $\mid \psi_{n_0}(t) \mid^2$ approaches the harmonic dependence Eq.(5) as it is seen in the insert in Fig. 3.

Spreading and tunnelling of the initial excitation is shown in Fig. 4. At the transient stage the excitation broadens up to the size of $2l(\bar{E}) \approx 80$, Fig. 4a. The initial stage is followed by the long-lasting stage of tunnelling at the macroscopic distance $2 \mid n_0 \mid$. The tunnelling gives rise to the secondary peak at $-n_0$, which "slowly" grows and reaches its maximum at $t \approx T/2$, Fig. 4c. The amplitude of the secondary peak in Figs. 3 and 4c does not exceed 5% of the initial peak, but at $t \approx T/2$ the both peaks have approximately the same amplitude. The amplitude of the peaks can be obtained from the normalization condition and it is determined by $l(E)$ as it follows from Eq. (5). This amplitude is much

larger then the amplitude at the origin $n = 0$, as it can be clearly seen from Fig. 4. The amplitude of the peaks at the points $\pm n_0$ increases with ϵ_0. Simultaneously the wave function at the origin decreases exponentially and can be easily controlled by the disorder.

Application of the proposed ideas to random electrical circuits is straightforward. In what follows we demonstrate the evolution of the signal in a circuit shown in Fig. 1 with vertical impedances being equal solenoids with $z_n = -i\omega L_0$ and horizontal impedances being capacitors with $Z_n = \frac{i}{\omega C_n} \approx \frac{i}{\omega C_0}\left(1 - \frac{\delta C_n}{C_0}\right)$. Here δC_n is the fluctuating part of the capacitance, which is an even function of n, $\delta C_n = \delta C_{-n}$. Propagation of an excitation in this circuit follows the wave equation

$$C_{n+1}\ddot{V}_{n+1} + C_{n-1}\ddot{V}_{n-1} - 2C_n\ddot{V}_n = V_n/L_0, \tag{6}$$

where V_n is voltage drop at the n-th capacitor. This equation requires two initial conditions. Let the voltage drop V_0 is applied at $t = 0$ to the capacitor C_{n_0}, inducing the initial current $I_0 = C_{n_0}\dot{V}_0$. Stationary solutions ($V_n \propto \exp(-i\omega t)$) are either even or odd functions and the spectrum of eigenfrequencies consists of a set of doublets. For an infinite chain the majority of the eigenfrequencies occupy an interval $[\omega_0/2, \infty]$, where $\omega_0 = (L_0C_0)^{-1/2}$. Assuming that the initial perturbation excites only the closest to the site n_0 pair of eigenstates (V_A and V_S), the solution of Eq. (6) can be written in the form similar to Eq. (5),

$$\begin{aligned} V_n(t) \approx \sqrt{\frac{1}{C_0 l(\bar{\omega})}} \Bigg\{ & \left[C_{n_0}V_0\sin(\bar{\omega}t) - \frac{I_0}{\bar{\omega}}\cos(\bar{\omega}t)\right]\sin\left(\frac{\Delta\omega}{2}t\right)V_-(n) \\ & + \left[C_{n_0}V_0\cos(\bar{\omega}t) + \frac{I_0}{\bar{\omega}}\sin(\bar{\omega}t)\right]\cos\left(\frac{\Delta\omega}{2}t\right)V_+(n)\Bigg\}. \end{aligned} \tag{7}$$

Here $\bar{\omega}$ is the center of the doublet, and $\Delta\omega$ is the frequency splitting. Single-peak functions $V_\pm(n) = (V_S \pm V_A)/\sqrt{2}$ play the same role as $\Psi_\pm$ do in Eq. (5). Eq. (7) shows that the evolution of the initial signal in a random (symmetric) electrical circuit is similar to the wave packet evolution obtained from the tight-binding model. There is an obvious symmetry-induced tunnelling of the initial signal at macroscopic distances.

The effect of tunnelling can be used for secure communications. Instead of coding and decoding a signal, we propose to suppress the transmitted signal by a circuit with random elements and then to restore it, using a symmetric counterpart of the random circuit. The signal can be suppressed to the noise level and safely transmitted to the receiver over a transmitting line. The symmetric counterpart of the random circuit restores only the signal,

(not noise) since constructive interference occurs only for the coherent part, which has passed trough the suppressing circuit of the emitter. The non-coherent part (noise or any irrelevant signal) will be exponentially suppressed by the receiving random circuit. The two identical random circuits may be fabricated as microchips, which are installed (or replaced) simultaneously at the emitter and receiver. In the absence of dissipation and asymmetry between the two random elements, the proposed method guarantees a robust restoration of the signal. Inevitable Joule losses should be compensated by an amplifier, which does not destroy the coherency of the signal.

In conclusion, we demonstrate that in a symmetric random potential the localized quantum states have two peaks as it is required by the parity. Fast spreading of the initial δ-excitation within localization length is followed by slow growth due to tunnelling at the symmetrical point. This effect opens a new possibility for secure communications that does not require coding and decoding of the transmitting signal. The random circuits may operate in a wide range of radio-frequencies, using commercial capacitors and inductors. They also can be fabricated and operated in the infrared and optical region using the concept of plasmonic nanoelements proposed in Ref.[16].

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[11] In a symmetric random potential the correlations always exist for the symmetric points, $\langle\epsilon_n\epsilon_{-n}\rangle = \epsilon_0^2$. At the same time the nearest sites may be uncorrelated. Because of this property the correlation function is not stationary. But it is stationary if the both sites are on the same side from the origin, $\langle\epsilon_n\epsilon_{n'}\rangle = \epsilon_0^2\xi(n-n')$, if $nn' > 0$.

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